# Lecture - 1

## Identical Particles.

Particles come in two* types : bosons and fermions. The wavefn of identical bosons is symmetric under particle exchange while that of fermions is anti-symmetric.

* This statement is strictly true only in three and higher spatial dimensions. In two dimensions, one can also have particles which pick up an arbitrary phase under exchange. In one dimension, bosons and fermions can be transformed into each other by a nonlocal transformation ('bosonization / fermionization').

Translating the above sentence into eqn, consider a wavefn of N bosons / fermions.

Identical Bosons :

$$\psi(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots x_N)$$

$$= \psi(x_1, x_2, \ldots x_j, \ldots x_i, \ldots x_N)$$

Identical fermions :

$$\psi(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots x_N)$$

$$= -\psi(x_1, x_2, \ldots x_j, \ldots x_i, \ldots x_N)$$

Let's study a simple example which illustrates this.

# Two Identical Bosons/Fermions in a Potential

$$\hat{H} = \frac{\hat{P}_1^2}{2m} + V(\hat{x}_1) + \frac{\hat{P}_2^2}{2m} + V(\hat{x}_2)$$

where $V(x)$ is some arbitrary potential. How does one find eigenstates of $\hat{H}$ given that the two particles are identical bosons / fermions?

Schrodinger's eqn:

$$\hat{H}\,\psi(x_1, x_2) = E\,\psi(x_1, x_2)$$

$$\Rightarrow \frac{-1}{2m}\frac{d^2\psi}{dx_1^2} - \frac{1}{2m}\frac{d^2\psi}{dx_2^2} + V(x_1)\psi + V(x_2)\psi = E\psi$$

The above differential equation is <u>separable</u> in the variables $x_1, x_2$ since there are no terms in $H$ that mix $x_1, x_2$.

This motivates the following ansatz:

$$\psi_{B/F} = u_i(x_1)\, u_j(x_2) \pm u_j(x_1)\, u_i(x_2)$$

where + sign refers to Bosons and − to fermions. Easy to check sym / anti-sym.

Claim: if $u_i$, $u_j$ are eigenfn's of the operator $\frac{p^2}{2m} + V$, then $\psi_{B/F}$ is an eigenfn of $\hat{H}$.

Proof: If $\left(\frac{p^2(x)}{2m} + V(x)\right) u_i(x) = E_i\, u_i(x)$

$$\left(\frac{p^2(x)}{2m} + V(x)\right) u_j(x) = E\, u_i(x)$$

then $\hat{H}\,\psi_{B/F} = \left[\frac{p_1^2}{2m} + V(x_1)\right]\left[u_i(x_1)\, u_j(x_2) \pm u_j(x_1)\, u_i(x_2)\right]$

$$+\left[\frac{p_2^2}{2m} + V(x_2)\right]\left[u_i(x_1)\, u_j(x_2) \pm u_j(x_1)\, u_i(x_2)\right]$$

$$= E_1 \, u_i(x_1) \, u_j(x_2)$$

$$\pm E_2 \, u_j(x_1) \, u_i(x_2)$$

$$+ E_2 \, u_i(x_1) \, u_j(x_2)$$

$$\pm E_1 \, u_j(x_1) \, u_i(x_2)$$

$$= (E_1 + E_2) \left[ u_i(x_1) \, u_j(x_2) \pm u_j(x_1) \, u_i(x_2) \right]$$

$$= (E_1 + E_2) \, \Psi_{B/F}(x_1, x_2). \qquad \text{QED.}$$

Pictorial Representation:

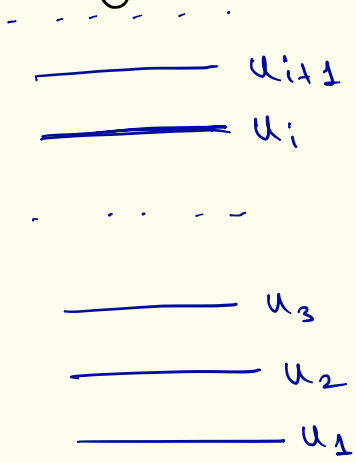
The Hamiltonian can be written as

$$\hat{H} = \hat{h}(x_1) + \hat{h}(x_2).$$

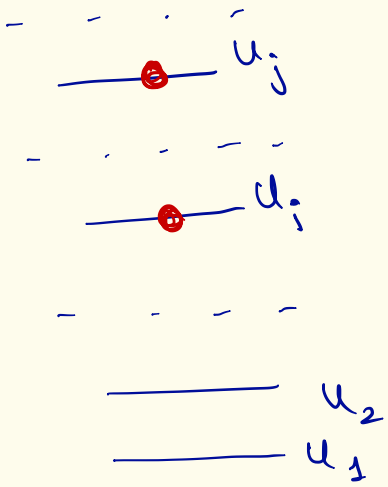
$\hat{h}(x)$ is called 'single particle Hamiltonian', and its eigenstates $u_n(x)$ are called 'single particle eigenstates'.

In contrast $\Psi_{B/F}(x_1, x_2)$ is called 'many-body eigenstate'.

Let's draw the single particle eigenenergies in increasing energy:




Since particles are identical, all one has to do specify the many body state $\Psi_{B/F}$ is to select the single particle levels that one is 'filling'. In the above example, we 'filled' the levels i and j.




Note that when $i = j$,

$$\psi_B(x_1, x_2) = u_i(x_1)\, u_i(x_2)$$

$$\psi_F(x_1, x_2) = 0$$

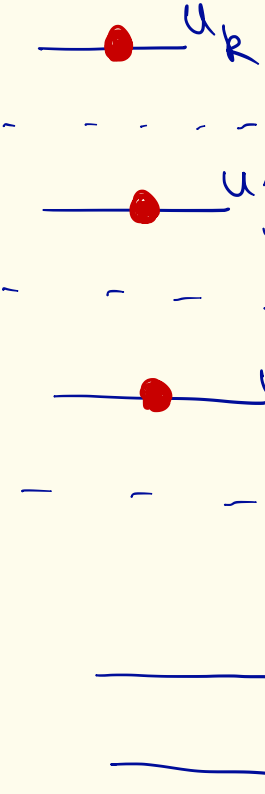
$\Rightarrow$ fermions must be in different states. This is called Pauli exclusion.

Now consider the case of three particles. Consider a many-body state obtained by filling levels $i, j, k$. Again, for

Such a state to exist for fermions, $i, j, k$ must be all distinct.

No such restriction for bosons.

Now, the picture looks like:




Given Energy of $\psi_{B/F}(x_1, x_2, x_3) = \epsilon_i + \epsilon_j + \epsilon_k$.

What is the wave-fn $\psi_{B/F}$ ?

Simple trick to write down the wavefn :

$$\psi_F = \det M$$

$$M = \begin{bmatrix} u_i(x_1) & u_i(x_2) & u_i(x_3) \\ u_j(x_1) & u_j(x_2) & u_j(x_3) \\ u_k(x_1) & u_k(x_2) & u_k(x_3) \end{bmatrix}$$

Interchanging $x_1 \leftrightarrow x_2 \Rightarrow$ interchanging columns of $M \Rightarrow \psi_F \rightarrow -\psi_F$ as required.

$\psi_B$ is obtained by changing all the negative signs to positive in the above wave-fn.

Explicitly,

$$\psi_B = u_i(x_1)\left[u_j(x_2)\,u_k(x_3) + u_j(x_3)\,u_k(x_2)\right]$$
$$+\; u_i(x_2)\left[u_j(x_1)\,u_k(x_3) + u_j(x_3)\,u_k(x_1)\right]$$
$$+\; u_i(x_3)\left[u_j(x_1)\,u_k(x_2) + u_j(x_2)\,u_k(x_1)\right]$$

This object is also sometimes denoted as 'permanent' of a matrix, as opposed to the determinant relevant for fermions.

Clearly the form of $\Psi_{B/F}$ becomes more and more tedious to write down explicitly as no. of particles increase ($N!$ terms). There must be a simpler way to write the ware-fn since we are just filling certain single-particle levels obtained by solving a quantum mechanics problem of <u>one</u> particle (the Hamiltonian $\hat{h} = \frac{p^2}{2m} + V$ above).

To set-up this notation, lets first work in a basis-independent way using bra-kets.

$\Psi$ B/F when single particle levels $|u_1\rangle, |u_2\rangle, \ldots |u_n\rangle$ are filled:

$$|\Psi_F\rangle = \frac{1}{\sqrt{n!}} \sum_P (-)^P |u_{p(1)}\rangle |u_{p(2)}\rangle \ldots |u_{p(n)}\rangle$$

where P runs over all the $n!$ permutations of the indices $1, 2, \ldots n$.

This is just rewriting the expression for determinant in a basis independent way. To see this, first we need to learn how to calculate the dot product of $|\Psi_F\rangle$ with a basis vector in real space, which is correctly symmetrized / anti-symmetrized.

i.e. define

$$|x_1, x_2, \cdots x_n\rangle_F$$

$$= \frac{1}{\sqrt{n!}} \sum_P (-)^P |x_{P(1)}\rangle |x_{P(2)}\rangle \cdots |x_{P(n)}\rangle$$

To obtain real-space wave-fn. we need to calculate the dot-product

$$\langle x_1, x_2, \cdots x_n | \psi_F \rangle$$

$$= \frac{1}{n!} \sum_{PQ} (-)^P (-)^Q \langle x_{Q(1)} | u_{P(1)} \rangle \langle x_{Q(2)} | u_{P(2)} \rangle \cdots \langle x_{Q(n)} | u_{P(n)} \rangle$$

$$= \frac{1}{n!} \sum_{PQ} (-)^{Q^{-1}P} \langle x_{(1)} | u_{Q^{-1}P(1)} \rangle \langle x_{(2)} | u_{Q^{-1}P(2)} \rangle \cdots \langle x_n | u_{Q^{-1}P(n)} \rangle$$

$$= \frac{1}{n!} \sum_P \sum_R (-)^R \langle x_{(1)} | u_{R(1)} \rangle \langle x_{(2)} | u_{R(2)} \rangle \cdots \langle x_{(n)} | u_{R(n)} \rangle$$

$$= \sum_R (-)^R \langle x_{(1)} | u_{R(1)} \rangle \langle x_{(2)} | u_{R(2)} \rangle$$
$$\cdots \langle x_{(n)} | u_{R(n)} \rangle$$

$$= \det \begin{bmatrix} u_1(x_1) & u_1(x_2) & \cdots & u_1(x_n) \\ u_2(x_1) & u_2(x_2) & \cdots & u_2(x_n) \\ \vdots & & & \\ u_n(x_1) & u_n(x_2) & \cdots & u_n(x_n) \end{bmatrix}$$

For bosons, one just drops the factor of $(-)^P$ everywhere, the rest of derivation is same.

Now we have already moved towards a notation that is faithful to the identical nature of particles and only focusses on the single-particle levels that are occupied by these particles.